Reinforcement design of concrete sections based on the arc-length method

Dimensionamento de seções de concreto armado baseado no processo do arco cilíndrico

Abstract

The reinforcement design of concrete cross-sections with the parabola-rectangle diagram is a well-established model. A global limit analysis, considering geometrical and material nonlinear behavior, demands a constitutive relationship that better describes concrete behavior. The Sargin curve from the CEB-FIP model code, which is defined from the modulus of elasticity at the origin and the peak point, represents the descending branch of the stress-strain relationship. This research presents a numerical method for the reinforcement design of concrete cross-sections based on the arc length process. This method is numerically efficient in the descending branch of the Sargin curve, where other processes present convergence problems. The examples discuss the reinforcement design of concrete sections based on the parabola-rectangle diagram and the Sargin curve using the design parameters of the local and global models, respectively.

Keywords: reinforced concrete, design of concrete cross-sections, Sargin curve, arc-length method.

Resumo

O dimensionamento de seções transversais de concreto com o diagrama parábola-retângulo é um modelo de cálculo consagrado. A análise limita global, considerando a não linearidade física e geométrica, demanda uma relação constitutiva que descreva melhor o comportamento do concreto. A curva de Sargin do Código Modelo CEB-FIP, que é definida a partir do módulo de elasticidade na origem e do ponto de pico, representa o ramo descendente da relação tensão-deformação. Esta pesquisa apresenta um método numérico de dimensionamento de seções transversais baseado no processo do arco-cilíndrico. Este método é numericamente eficiente no ramo descendente da curva de Sargin, onde outros processos mostram problemas de convergência. Os exemplos discutem o dimensionamento de seções transversais com o diagrama parábola-retângulo e a curva de Sargin, utilizando os parâmetros de cálculo dos modelos local e global, respectivamente.

Palavras-chave: concreto armado, dimensionamento de seções de concreto, curva de Sargin, processo do arco-cilíndrico.

© 2018 IBRACON
1. Introduction

Different constitutive relations have been used for the reinforcement design of concrete beams and columns. Mörsch [1] considered linear-elastic material behavior in allowable stress design. Several authors contributed to the flexural model that is nowadays used in ultimate limit state design. In the 1950s, Bernoulli’s plate section hypothesis, equilibrium conditions, and nonlinear constitutive relationships for concrete and steel provided the basis for the development of reinforcement design theories. The literature review on concrete stress distribution presented by Hognestad [2] includes the contributions of Whitney [3] and Bittner [4] for rectangular and parabola-rectangle diagrams, respectively.

Simplified theories for ultimate strength under combined bending and normal force were consolidated in the early 1960s using approximate constitutive relations for concrete without any significant loss of precision. Mattock, Kriz, and Hognestad [5] adopted the rectangular diagram, while Rüsch, Grasser, and Rao [6] used the parabola-rectangle diagram.


Such simplified stress diagrams require limiting strain states for reinforcing steel and concrete to ensure valid results under combined axial and bending effects. The approximated diagrams simplify numerical design procedures and design graphs, but do not represent the characteristics of concrete, such as the initial modulus of elasticity and the descending branch of the stress-strain relationship.

Physical and geometric nonlinear analyses of reinforced concrete framed structures require stress-strain relationships that better describe the behavior of concrete. The Sargin model [11] represents several characteristics of the uniaxial behavior of concrete. The Sargin curve presented in the CEB-FIP Model Code 1990 [12] is defined by the initial modulus of elasticity, minimum compression stress, and critical strain. This curve also represents the descending branch of the stress-strain relationship.

The convergence of the Newton-Raphson method is not stable in descending branches of stress-strain curves. This study presents a numerical method for the reinforcement design of concrete sections under combined bending and normal forces that is suitable for the Sargin curve. It is based on the arc-length technique, which is stable for negative derivatives of the stress-strain diagram. The numerical procedure automatically identifies the strain distribution in the ultimate limit state without having to consider a variable strain limit in compression (domain 5). Concrete and steel strain limits are not required but can be included to avoid excessive deformations. The examples given of reinforcement design apply both the parabola-rectangle and the Sargin curve. Design stress-strain diagrams are based on characteristic curves and code provisions for local and global analysis.

2. Simplifying assumptions

The following assumptions are considered at the outset:
1. There is no relative displacement between the steel and the surrounding concrete (steel and concrete have the same mean strain).

2. Cross-sections remain plane after deformation (Bernoulli’s hypothesis). In the interests of simplifying the formulation, steel area is not deducted from concrete area. The influence of the type of aggregate is not discussed in the present investigation.

3. Constitutive relations

Compression stresses and strains are negative. The constitutive stress-strain relationship of steel is defined by

\[ \sigma_s = \sigma_s(\varepsilon_s) \]

where steel stress \( \sigma_s \) is a function of steel strain \( \varepsilon_s \). The yield strength and modulus of elasticity of the steel are \( f_y \) and \( E_s \), respectively. The corresponding yield strain \( \varepsilon_{sy} \) is:

\[ \varepsilon_{sy} = \frac{f_y}{E_s} \]

The steel stress-strain curve is divided into three regions (Figure 1), which are respectively defined by:

\[ \sigma_s = -f_y + K_s E_s (\varepsilon_s + \varepsilon_{sy}) \quad \text{for} \quad \varepsilon_s < -\varepsilon_{sy} \]

\[ \sigma_s = E_s \varepsilon_s \quad \text{for} \quad -\varepsilon_{sy} < \varepsilon_s < \varepsilon_{sy} \]

\[ \sigma_s = f_y + K_s E_s (\varepsilon_s - \varepsilon_{sy}) \quad \text{for} \quad \varepsilon_s > \varepsilon_{sy} \]

The convergence of the Newton-Raphson process in the yielding range is stabilized by the reduced tangent modulus \( K_s E_s \). The arc-length method uses \( K = 0 \). The steel tangent modulus \( E_s(\varepsilon_s) \) is defined by the derivative:

\[ E_s(\varepsilon_s) = \frac{\partial \sigma_s}{\partial \varepsilon_s} \]

Expressions and yield:

\[ E_s(\varepsilon_s) = K_s E_s \quad \text{for} \quad \varepsilon_s < -\varepsilon_{sy} \]

\[ E_s(\varepsilon_s) = E_s \quad \text{for} \quad -\varepsilon_{sy} < \varepsilon_s < \varepsilon_{sy} \]

\[ E_s(\varepsilon_s) = K_s E_s \quad \text{for} \quad \varepsilon_s > \varepsilon_{sy} \]

Concrete stress \( \sigma_c \) is a function of concrete strain \( \varepsilon_c \), i.e.,

\[ \sigma_c = \sigma_c(\varepsilon_c) \]
CEB-FIP Model Code 1990 [12] defines the Sargin curve from the minimum compression stress \( \sigma_{c1} \), the critical strain \( \varepsilon_{c1} \), and the initial modulus of elasticity \( E_{c0} \) (Figure 2). Concrete stress is defined by:

\[
\varepsilon_c = \begin{cases} \frac{1}{\beta_{c}+\gamma_{c}} & \text{for } \varepsilon_c \leq \varepsilon_{c,\text{lim}} \\ \frac{1}{(1-2)\beta_{c}+1} & \text{for } \varepsilon_{c,\text{lim}} \leq \varepsilon_c \leq 0 \\ 0 & \text{for } 0 \leq \varepsilon_c \\
\end{cases}
\]

(7)

where \( \varepsilon_{c,\text{lim}} \) is the strain that separates the first two branches of the curve. The secant modulus of elasticity \( E_{c1} \) at the critical point is:

\[
E_{c1} = \frac{\sigma_{c1}}{\varepsilon_{c1}}
\]

Coefficient \( k_1 \), variable \( \eta \), and strain limit \( \varepsilon_{c,\text{lim}} \) are respectively defined by:

\[
k_1 = \frac{E_{c0}}{E_{c1}}
\]

\[
\eta = \frac{\varepsilon_{c}}{\varepsilon_{c1}}
\]

\[
\varepsilon_{c,\text{lim}} = \eta_{\text{lim}} \varepsilon_{c1}
\]

where

\[
\eta_{\text{lim}} = k_2 + \sqrt{k_2^2 - \frac{1}{2}}
\]

(12)

\[
k_2 = \frac{1}{2} \left( \frac{k_1}{2} + 1 \right)
\]

Parameters \( b \) and \( c \) of equation (7) are respectively expressed by:

\[
b = \frac{\varepsilon_{c,\text{lim}} - 2}{\eta_{\text{lim}}}
\]

\[
c = \frac{4}{\eta_{\text{lim}} - \varepsilon_{c,\text{lim}}}
\]

(14)

(15)

The tangent modulus of elasticity of concrete, \( E_{c} (\varepsilon_{c}) \), is defined by the derivative:

\[
E_{c}(\varepsilon_{c}) = \frac{\partial \varepsilon_{c}}{\partial \varepsilon_{c}}
\]

(17)

Expressions and yield:

\[
E_{c}(\varepsilon_{c}) = \frac{E_{c0} (\varepsilon_{c} + 2\eta_{\text{lim}})}{(\eta_{\text{lim}} + \gamma_{c})}
\]

(8)

\[
E_{c}(\varepsilon_{c}) = \frac{\gamma_{c}}{(\eta_{\text{lim}} + \gamma_{c})^2}
\]

(9)

\[
E_{c}(\varepsilon_{c}) = 0
\]

(10)

The initial modulus of elasticity can be ascertained from equations 8 and 9, i.e.,

\[
E_{c}(0) = E_{c0} k_1 = E_{c0}
\]

(11)

The provisions of item 5.8.6 from CEN Eurocode 2:2004 [8] are also considered. The critical strain and initial elasticity modulus are, respectively,

\[
\varepsilon_{c1} = -0.7 f_{cm} \left( \varepsilon_{c1} \right)^{0.31} / 1000 \geq -0.0028 \quad f_{cm} [\text{MPa}]
\]

(20)

\[
E_{c0} = (1.05) (22000) (f_{cm}/10)^{0.3} / \gamma_{c} \quad E_{c0} [f_{cm} [\text{MPa}]]
\]

(21)

The partial factor for the elasticity modulus of concrete is \( \gamma_{c} = 1.2 \) and the effect of the aggregate type is not discussed in this investigation. The mean compressive strength of the concrete is estimated by \( f_{cm} = f_{ck} + 8 \text{ MPa} \), where \( f_{ck} \) is the characteristic compressive strength of concrete.

The partial safety factors for concrete and steel are \( \gamma_{c} = 1.4 \) and \( \gamma_{s} = 1.15 \), respectively, as recommended in ABNT NBR 6118:2014 [10]. The effect of long-term sustained loads on the ultimate strength of concrete (Rüsch [13]) is considered by using \( \varepsilon_c = 0.85 \) in:

\[
\sigma_{c} = -\alpha_{c} f_{ck} / \gamma_{c}
\]

(22)

The reinforcement design examples apply both the Sargin and the parabola-rectangle curve. The reinforcement design with the parabola-rectangle diagram assumes the constitutive relation, the limit strains, and the ultimate limit-state domains provided in ABNT NBR 6118:2014 [10].

The numerical procedure proposed for the Sargin curve automatically identifies the strain distribution in the ultimate limit state without having to consider a variable strain limit in compression (domain 5). Concrete and steel strain limits are not required, but they are included to avoid excessive deformations. Steel strain is limited by:

\[
\varepsilon_{s} \leq 0.010
\]

(23)

Concrete strain is limited by:

\[
\varepsilon_{c} \geq \varepsilon_{c1}
\]

(24)

CEN Eurocode 2:2004 [8] provides the following expression:

\[
\varepsilon_{c1} = -0.0028 - 0.027 \left( (98 - f_{ck}) / 100 \right)^{4} \geq -0.0035 \quad f_{cm} [\text{MPa}]
\]

(25)

![Figure 2: Stress-strain relationship of concrete](image-url)
Since $\varepsilon_{c u1} > \varepsilon_{c \text{lim}}$ (Figure 2), the branch of the Sargin curve defined by $\varepsilon_{c} \leq \varepsilon_{c \text{lim}}$ is not used in the reinforcement design.

4. Equilibrium and compatibility equations

Figure 3 shows the coordinate system of the cross-section. The concrete section is discretized into area elements $dA_c$. The position of each element centroid is defined by the coordinates $y_c$ and $z_c$. The position of each steel reinforcing bar, whose area is denoted as $A_s$, is defined by the coordinates $y_s$ and $z_s$ (Figure 4). The stress resultants are presented in Figure 5. Positive normal forces $N_x$ are tension forces. Positive bending moments $M_y$ and $M_z$ correspond to tension stresses at the positive y and z faces, respectively.

According to assumption 1, there is no slip between the steel and the surrounding concrete. Concrete and steel strains, which are respectively denoted as $\varepsilon_c$ and $\varepsilon_s$, have the same value, i.e.,

$$\varepsilon_c = \varepsilon_s = \varepsilon$$  \hspace{1cm} (26)

where $\varepsilon$ is the strain at a point in the cross-section.

Cross-sections remain plane after deformation (assumption 2). Strain $\varepsilon$ at a point is expressed as:

$$\varepsilon = k_x y + k_z z$$  \hspace{1cm} (27)

where $k_x$ is the strain at the origin. Parameters $k_x$ and $k_z$ are the curvatures with inverted signs. The compatibility equation (27) is rewritten as:

$$\varepsilon = p^T k$$  \hspace{1cm} (28)

where $p = [1 \ y \ z]^T$ is a position vector and $k = [k_x \ k_y \ k_z]^T$ is the generalized strain vector.

The following expressions are obtained from the equilibrium conditions of the cross-section:

$$N_x = \int_A \sigma_x dA_c + \sum \sigma_x A_s$$  \hspace{1cm} (29)

$$M_y = \int_A \sigma_y y dA_c + \sum \sigma_y y A_s$$  \hspace{1cm} (30)

$$M_z = \int_A \sigma_z z dA_c + \sum \sigma_z z A_s$$  \hspace{1cm} (31)

The equilibrium equations (29), (30), and (31) are rewritten as:

$$S = \int_A p \sigma(\varepsilon) dA$$  \hspace{1cm} (32)

where $\sigma(\varepsilon)$ is the stress at a point and $S = [N_x \ M_y \ M_z]^T$ is the stress resultant vector. The following incremental equation is obtained from (32):
E(\(\varepsilon\)) is the tangent modulus of elasticity at a point. The substitution of (28) into (33) yields:

\[
\Delta S = \int_A p \, \Delta \sigma(e) \, dA = \int_A p \, E(e) \, \Delta \varepsilon \, dA
\]

where the tangent matrix \(E\) is expressed by:

\[
E = \int_A p \, E(e) \, p^T \, dA
\]

5. Numerical methods for section analysis and reinforcement design

Figure 6 shows the solution for a nonlinear structural system of a single degree of freedom based on the Newton-Raphson process. The arc-length process is a variant of the Newton-Raphson method that controls the progress of the iterative process (Figure 7). The arc-length and load factor are denoted as \(l\) and \(\lambda\), respectively. The incremental process is capable of passing through critical points.

The section analysis and reinforcement design methods are applicable, but not limited, to the Sargin stress-strain relationship.

5.1 Arc-length method

The arc-length method presented by Crisfield [14] is an alternative formulation of the method originally proposed by Riks [15]. The stress resultant vector is defined as \(\lambda S\), where \(\lambda\) is a load factor and \(S = [N, M_x, M_y, M_z]\) is the stress resultant vector that is established as a reference. The term \(\Delta S_i\) is defined as:

\[
\Delta S_i = \lambda S - S_i
\]

where \(S_i = [N_{x,i}, M_{x,i}, M_{y,i}]\) is the stress resultant vector associated with the generalized strain vector \(k_i = [k_{x,i}, k_{y,i}, k_{z,i}]^T\) at iteration \(i\). The term \(\Delta S_i\) is written as:

\[
\Delta k_i = E_i^{-1} \Delta S_i
\]

where \(E_i\) is a tangent matrix and \(\Delta k_i\) is the increment of the generalized strain vector at iteration \(i\). Equations (36) and (37) yield:

\[
\Delta k_i = \lambda E_i^{-1} S - E_i^{-1} S_i = \lambda g_i - g_i
\]

Expression (42) defines the quadratic equation:

\[
ax^2 + bx + c = 0
\]

where

\[
a = \Delta S_i \cdot \Delta S_i^T ; \quad b = -2 \Delta S_i \cdot \Delta k_i ; \quad c = \Delta k_i \cdot \Delta k_i
\]

One of the roots of equation (43) corresponds to the factor \(\lambda\) of the next iteration. The appropriate root is discussed in the next item.

5.2 Section analysis

The parameters required for section analysis are the steel and concrete properties, the geometric characteristics of the cross-section, the position and area of the reinforcing steel bars, the reference stress resultant vector \(\bar{S}\), and the arc-length \(l\). The maximum load factor \(\bar{\lambda}\) found throughout the incremental process defines the cross-section strength.

A brief summary of the iterative process is presented next.

I. Generalized strains \(k_i\) at iteration \(i\)

Iteration \(i\) starts with vector \(k_i\). The first iteration can start with \(k_i = 0\).
II. Generalized stresses $S_i$ and tangent matrix $E_i$

The strains $\varepsilon = \mathbf{p}^T k$, stresses $\sigma(\varepsilon)$, and tangent moduli of elasticity $E(\varepsilon)$ are determined for each area element of the steel and concrete. Expressions (32) and (35) yield the generalized stresses $S_i$ and tangent matrices $E_i$, respectively.

III. Load factors $\lambda_a$ and $\lambda_b$

Equations (39) and (40) yield the auxiliary vectors $\mathbf{g}_a$ and $\mathbf{g}_b$, respectively. Load factors $\lambda_a$ and $\lambda_b$ are the solutions of the quadratic equation established by (43) and (44).

IV. Load factor $\lambda$

The root of (43) that pushes forward the incremental process is selected. The first iteration elects $\lambda_1 = \max (\lambda_a, \lambda_b)$. For iteration $i>1$, equation (38) yields:

$$\Delta k_A = \lambda_A \mathbf{g}_A - \mathbf{g}$$  \hspace{1cm} (45)

$$\Delta k_B = \lambda_B \mathbf{g}_B - \mathbf{g}$$  \hspace{1cm} (46)

where $\Delta k_A$ and $\Delta k_B$ are the strain vector increments of roots $\lambda_A$ and $\lambda_B$, respectively.

The slopes $\theta_A$ and $\theta_B$ of roots $\lambda_A$ and $\lambda_B$ are respectively defined as:

$$\theta_A = \Delta k_A^T \Delta k_A$$  \hspace{1cm} (47)

$$\theta_B = \Delta k_B^T \Delta k_B$$  \hspace{1cm} (48)

The load factor $\lambda$ associated with the maximum slope $\theta = \max (\theta_A, \theta_B)$ is selected. The corresponding increment $\Delta k$ of the next iteration is:

$$k_{i+1} = k_i + \Delta k$$  \hspace{1cm} (49)

The procedure returns to step II to start a new iteration. The process terminates when steel or concrete strains reach their limit values. Section strength is defined by $\overline{\lambda} S$, where $\overline{\lambda}$ is the maximum load factor found throughout the incremental process.

5.3 Reinforcement design

The parameters required for reinforcement design are the steel and concrete properties, the geometric characteristics of the cross-section, the position and relative area of each reinforcing steel bar, the minimum and maximum steel ratios, the reference stress resultant vector $\bar{S}$, and the arc-length $l$. The design stress resultants are defined by $\lambda_s S$, where $\lambda_s$ is the corresponding load factor. A brief summary of the iterative process is presented next.

II. Stress analysis for maximum reinforcement

The procedure in item 5.2 yields the maximum load factor $\lambda_{sAmax}$ for the maximum reinforcement $A_{sAmax}$. If $\lambda_s > \lambda_{sAmax}$, the cross-section is not adequate and the process is terminated. Otherwise, $\lambda_{sINF} = \lambda_{sAmax}$ and $A_{sINF} = A_{sAmax}$.

III. Iterative process

The required reinforcement is estimated by linear interpolation

$$A_s = A_{sINF} + \frac{(A_{sSUP} - A_{sINF})}{\lambda_{sSUP} - \lambda_{sINF}} \lambda_d$$  \hspace{1cm} (50)

The procedure in item 5.2 yields the maximum load factor $\lambda$ for $A_s$. If $\lambda > \lambda_d$, the new limit is defined by $\lambda_{SUP} = \lambda$ and $A_{SUP} = A_s$. Otherwise, $\lambda_{INF} = \lambda$ and $A_{INF} = A_s$.

A new iteration restarts when $A_{sSUP} - A_{sINF} > TOL_s$, where $TOL_s$ is the tolerance for the reinforcement design. The iterative process ends when $A_{sSUP} - A_{sINF} \leq TOL_s$. The required reinforcement is conservatively assumed to be $A_{sSUP}$. This study considers $TOL_s = 1 \times 10^{-7}$ m².

6. Examples and numerical results

The reinforcement design procedure based on the arc-length method is implemented in two Fortran programs, which use parabola-rectangle and Sargin curves, respectively. Programs Fx4 and Fx5 are presented in Kabenjabu [16].

The typical rectangular cross-section is defined by $b_y = 0.25$ m and $b_z = 0.80$ m (Figure 8). The rebar edge distances in $y$ and $z$ directions are $d'_y = 0.05$ m and $d'_z = 0.05$ m, respectively.

The concrete section is discretized in $25 \times 80$ area elements. The section is considered doubly reinforced in most examples, but it is also studied as singly reinforced for pure bending.

The characteristic yield strength of steel is $f_{y\mu} = 500$MPa.

Figure 8

Typical cross-sections with and without compression reinforcement
The examples investigate concrete grades C15, C30, and C45. The corresponding compressive strengths are 15 MPa, 30 MPa and 45 MPa, respectively. Although C15 concrete is no longer in use, it is included in the study because of its widespread use in the past.

The partial safety factors for concrete and steel are $\gamma_c = 1.4$ and $\gamma_s = 1.15$, respectively, as recommended in ABNT NBR 6118:2014 [10]. $N_x$, $M_y$, and $M_z$ are the design values of the stress resultants. The examples are summarized in Tables 1 to 9, where $A_s$ is the required total reinforcement, $\varepsilon_{c_{\text{min}}}$ is the minimum concrete strain, and $\varepsilon_{s_{\text{max}}}$ is the maximum steel strain. The relative difference $\Delta A_s/\Delta A_{\text{tot}}$ is defined as:

$$\Delta A_s/\Delta A_{\text{tot}} = (A_s - A_{\text{tot}})/A_{\text{tot}}$$  \hspace{1cm} (51)

where $A_s$ and $A_{\text{tot}}$ are the required total reinforcement for parabola-rectangle and Sargin curves, respectively.

The section is subjected to pure compression in Table 1. The Sargin curve yields lower reinforcement values than the parabola-rectangle diagram. The limit strain $\varepsilon_{c_{\text{crit}}}$ is -0.002 of the parabola-rectangle curve.

### Table 1
Doubly-reinforced cross-section subjected to pure compression

<table>
<thead>
<tr>
<th>Stress resultants</th>
<th>Parabola-rectangle diagram</th>
<th>Sargin curve</th>
<th>Diff.</th>
<th>Relative difference</th>
</tr>
</thead>
<tbody>
<tr>
<td>$N_x$ (kN)</td>
<td>$M_y$ (kNm)</td>
<td>$M_z$ (kNm)</td>
<td>$A_{s_{\text{tot}}}$ (cm²)</td>
<td>$\varepsilon_{c_{\text{min}}}$</td>
</tr>
<tr>
<td>C15</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>-3000 0 0 28.0 -0.00200 -0.00200 27.2 -0.00207 -0.00207 -0.8 -2.9%</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>-4000 0 0 51.8 -0.00200 -0.00200 50.2 -0.00207 -0.00207 -1.6 -3.1%</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>-5000 0 0 75.6 -0.00200 -0.00200 73.2 -0.00207 -0.00207 -2.4 -3.2%</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>C30</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>-4000 0 0 8.5 -0.00200 -0.00200 8.2 -0.00216 -0.00216 -0.3 -3.4%</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>-5000 0 0 23.3 -0.00200 -0.00200 21.9 -0.00216 -0.00216 -1.1 -3.4%</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>-6000 0 0 44.2 -0.00200 -0.00200 42.7 -0.00216 -0.00216 -1.5 -3.4%</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>-7000 0 0 79.9 -0.00200 -0.00200 77.2 -0.00216 -0.00216 -2.1 -3.4%</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>C45</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>-6000 0 0 12.7 -0.00200 -0.00200 12.4 -0.00240 -0.00240 -0.4 -3.3%</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>-7000 0 0 36.5 -0.00200 -0.00200 35.3 -0.00240 -0.00240 -1.2 -3.3%</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>-8000 0 0 60.3 -0.00200 -0.00200 58.4 -0.00240 -0.00240 -2.0 -3.3%</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

### Table 2
Doubly-reinforced cross-section subjected to compression and uniaxial bending ($e_z = b/4$)

<table>
<thead>
<tr>
<th>Stress resultants</th>
<th>Parabola-rectangle diagram</th>
<th>Sargin curve</th>
<th>Diff.</th>
<th>Relative difference</th>
</tr>
</thead>
<tbody>
<tr>
<td>$N_x$ (kN)</td>
<td>$M_y$ (kNm)</td>
<td>$M_z$ (kNm)</td>
<td>$A_{s_{\text{tot}}}$ (cm²)</td>
<td>$\varepsilon_{c_{\text{min}}}$</td>
</tr>
<tr>
<td>C15</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>-2250 0 450 9.7 -0.00350 0.00103 8.4 -0.00295 0.00085 0.1 1.3%</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>-2500 0 500 24.4 -0.00350 0.00077 23.1 -0.00284 0.00034 0.0 0.0%</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>-2750 0 550 59.4 -0.00350 -0.00008 58.5 -0.00270 -0.00001 -0.2 -0.3%</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>-3000 0 600 77.3 -0.00339 -0.00016 76.5 -0.00266 -0.00009 -0.2 -0.3%</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>C30</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>-2250 0 450 9.7 -0.00350 0.00103 8.4 -0.00295 0.00085 0.1 1.3%</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>-2500 0 500 24.4 -0.00350 0.00077 23.1 -0.00284 0.00034 0.0 0.0%</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>-2750 0 550 59.4 -0.00350 -0.00008 58.5 -0.00270 -0.00001 -0.2 -0.3%</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>-3000 0 600 77.3 -0.00339 -0.00016 76.5 -0.00266 -0.00009 -0.2 -0.3%</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>C45</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>-2250 0 450 9.7 -0.00350 0.00103 8.4 -0.00295 0.00085 0.1 1.3%</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>-2500 0 500 24.4 -0.00350 0.00077 23.1 -0.00284 0.00034 0.0 0.0%</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>-2750 0 550 59.4 -0.00350 -0.00008 58.5 -0.00270 -0.00001 -0.2 -0.3%</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>-3000 0 600 77.3 -0.00339 -0.00016 76.5 -0.00266 -0.00009 -0.2 -0.3%</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
### Table 3
Doubly-reinforced cross-section subjected to compression and uniaxial bending ($\varepsilon_\text{y} = b_y/2$)

<table>
<thead>
<tr>
<th>Stress resultants</th>
<th>Parabola-rectangle diagram</th>
<th>Sargin curve</th>
<th>Diff.</th>
<th>Relative difference</th>
</tr>
</thead>
<tbody>
<tr>
<td>$N$ (kN)</td>
<td>$M_y$ (kNm)</td>
<td>$M_z$ (kNm)</td>
<td>$A_{\text{tot}}$ (cm²)</td>
<td>$\varepsilon_{c\min}$</td>
</tr>
<tr>
<td>-750</td>
<td>0</td>
<td>300</td>
<td>8.3</td>
<td>-0.00350</td>
</tr>
<tr>
<td>-1000</td>
<td>0</td>
<td>400</td>
<td>16.1</td>
<td>-0.00350</td>
</tr>
<tr>
<td>-1250</td>
<td>0</td>
<td>500</td>
<td>26.7</td>
<td>-0.00351</td>
</tr>
<tr>
<td>-1500</td>
<td>0</td>
<td>600</td>
<td>38.0</td>
<td>-0.00350</td>
</tr>
<tr>
<td>-1750</td>
<td>0</td>
<td>700</td>
<td>49.7</td>
<td>-0.00350</td>
</tr>
<tr>
<td>-2000</td>
<td>0</td>
<td>800</td>
<td>61.6</td>
<td>-0.00350</td>
</tr>
</tbody>
</table>

### Table 4
Doubly-reinforced cross-section subjected to compression and biaxial bending ($\varepsilon_\text{y} = b_y/4$ and $\varepsilon_\text{z} = b_z/4$)

<table>
<thead>
<tr>
<th>Stress resultants</th>
<th>Parabola-rectangle diagram</th>
<th>Sargin curve</th>
<th>Diff.</th>
<th>Relative difference</th>
</tr>
</thead>
<tbody>
<tr>
<td>$N$ (kN)</td>
<td>$M_y$ (kNm)</td>
<td>$M_z$ (kNm)</td>
<td>$A_{\text{tot}}$ (cm²)</td>
<td>$\varepsilon_{c\min}$</td>
</tr>
<tr>
<td>-750</td>
<td>0</td>
<td>300</td>
<td>8.3</td>
<td>-0.00350</td>
</tr>
<tr>
<td>-1000</td>
<td>0</td>
<td>400</td>
<td>16.1</td>
<td>-0.00350</td>
</tr>
<tr>
<td>-1250</td>
<td>0</td>
<td>500</td>
<td>26.7</td>
<td>-0.00351</td>
</tr>
<tr>
<td>-1500</td>
<td>0</td>
<td>600</td>
<td>38.0</td>
<td>-0.00350</td>
</tr>
<tr>
<td>-1750</td>
<td>0</td>
<td>700</td>
<td>49.7</td>
<td>-0.00350</td>
</tr>
<tr>
<td>-2000</td>
<td>0</td>
<td>800</td>
<td>61.6</td>
<td>-0.00350</td>
</tr>
</tbody>
</table>

The relative differences are always less than 5% in Tables 3, 5, 7 and 8.
### Table 5
Doubly-reinforced cross-section subjected to compression and biaxial bending ($e_y = b_y/2$ and $e_z = b_z/2$)

<table>
<thead>
<tr>
<th>Stress resultants</th>
<th>Parabola-rectangle diagram</th>
<th>Sargin curve</th>
<th>Diff.</th>
<th>Relative difference</th>
</tr>
</thead>
<tbody>
<tr>
<td>$N_x$ (kN)</td>
<td>$M_y$ (kNm)</td>
<td>$M_z$ (kNm)</td>
<td>$A_{tot}$ (cm$^2$)</td>
<td>$\varepsilon_{c_{min}}$</td>
</tr>
<tr>
<td>-400 50 160</td>
<td>10.3 -0.00350 0.00364</td>
<td>9.8 -0.00350 0.00407</td>
<td>-0.5</td>
<td>-4.8%</td>
</tr>
<tr>
<td>-600 75 240</td>
<td>21.2 -0.00350 0.00268</td>
<td>20.6 -0.00350 0.00300</td>
<td>-0.6</td>
<td>-2.7%</td>
</tr>
<tr>
<td>-800 100 320</td>
<td>33.2 -0.00350 0.00216</td>
<td>32.7 -0.00350 0.00232</td>
<td>-0.5</td>
<td>-1.5%</td>
</tr>
<tr>
<td>-1000 125 400</td>
<td>47.6 -0.00350 0.00191</td>
<td>45.6 -0.00350 0.00205</td>
<td>-2.0</td>
<td>-4.3%</td>
</tr>
<tr>
<td>-1200 150 480</td>
<td>63.1 -0.00350 0.00177</td>
<td>61.0 -0.00350 0.00189</td>
<td>-2.1</td>
<td>-3.3%</td>
</tr>
<tr>
<td>-600 75 240</td>
<td>11.7 -0.00350 0.00456</td>
<td>11.4 -0.00350 0.00477</td>
<td>-0.2</td>
<td>-2.0%</td>
</tr>
<tr>
<td>-800 100 320</td>
<td>20.7 -0.00350 0.00364</td>
<td>20.4 -0.00350 0.00381</td>
<td>-0.3</td>
<td>-1.4%</td>
</tr>
<tr>
<td>-1000 125 400</td>
<td>31.0 -0.00350 0.00308</td>
<td>30.8 -0.00350 0.00323</td>
<td>-0.3</td>
<td>-0.9%</td>
</tr>
<tr>
<td>-1200 150 480</td>
<td>42.3 -0.00350 0.00268</td>
<td>42.0 -0.00350 0.00281</td>
<td>-0.3</td>
<td>-0.6%</td>
</tr>
<tr>
<td>-1400 175 560</td>
<td>54.1 -0.00350 0.00239</td>
<td>53.9 -0.00350 0.00251</td>
<td>-0.2</td>
<td>-0.4%</td>
</tr>
<tr>
<td>-1600 200 640</td>
<td>66.3 -0.00350 0.00216</td>
<td>66.2 -0.00350 0.00226</td>
<td>-0.1</td>
<td>-0.2%</td>
</tr>
<tr>
<td>-400 50 160</td>
<td>12.2 -0.00350 0.00634</td>
<td>12.1 -0.00350 0.00630</td>
<td>0.1</td>
<td>1.8%</td>
</tr>
<tr>
<td>-600 75 240</td>
<td>22.0 -0.00350 0.00500</td>
<td>21.9 -0.00350 0.00498</td>
<td>0.3</td>
<td>1.9%</td>
</tr>
<tr>
<td>-800 100 320</td>
<td>31.0 -0.00350 0.00364</td>
<td>30.9 -0.00350 0.00363</td>
<td>0.5</td>
<td>1.6%</td>
</tr>
<tr>
<td>-1000 125 400</td>
<td>41.2 -0.00350 0.00324</td>
<td>41.1 -0.00350 0.00323</td>
<td>0.6</td>
<td>1.4%</td>
</tr>
<tr>
<td>-1200 150 480</td>
<td>52.1 -0.00350 0.00293</td>
<td>52.1 -0.00350 0.00292</td>
<td>0.7</td>
<td>1.3%</td>
</tr>
<tr>
<td>-1400 175 560</td>
<td>63.5 -0.00350 0.00268</td>
<td>63.6 -0.00350 0.00267</td>
<td>0.8</td>
<td>1.2%</td>
</tr>
</tbody>
</table>

### Table 6
Doubly-reinforced cross-section subjected to compression and uniaxial bending ($e_y = b_y/4$)

<table>
<thead>
<tr>
<th>Stress resultants</th>
<th>Parabola-rectangle diagram</th>
<th>Sargin curve</th>
<th>Diff.</th>
<th>Relative difference</th>
</tr>
</thead>
<tbody>
<tr>
<td>$N_x$ (kN)</td>
<td>$M_y$ (kNm)</td>
<td>$M_z$ (kNm)</td>
<td>$A_{tot}$ (cm$^2$)</td>
<td>$\varepsilon_{c_{min}}$</td>
</tr>
<tr>
<td>-1250 78.125 0</td>
<td>12.2 -0.00350 0.00048</td>
<td>12.3 -0.00286 0.00056</td>
<td>0.1</td>
<td>1.2%</td>
</tr>
<tr>
<td>-1500 93.750 0</td>
<td>22.0 -0.00350 0.00031</td>
<td>22.1 -0.00280 0.00037</td>
<td>0.1</td>
<td>0.3%</td>
</tr>
<tr>
<td>-1750 109.375 0</td>
<td>32.2 -0.00350 0.00021</td>
<td>32.2 -0.00277 0.00027</td>
<td>0.0</td>
<td>0.0%</td>
</tr>
<tr>
<td>-2000 125.000 0</td>
<td>42.5 -0.00350 0.00014</td>
<td>42.4 -0.00275 0.00020</td>
<td>-0.1</td>
<td>-0.1%</td>
</tr>
<tr>
<td>-2250 140.625 0</td>
<td>52.8 -0.00350 0.00010</td>
<td>52.8 -0.00274 0.00015</td>
<td>-0.1</td>
<td>-0.1%</td>
</tr>
<tr>
<td>-2500 156.250 0</td>
<td>63.3 -0.00350 0.00006</td>
<td>63.2 -0.00273 0.00011</td>
<td>-0.1</td>
<td>-0.2%</td>
</tr>
<tr>
<td>-2750 171.875 0</td>
<td>73.7 -0.00350 0.00004</td>
<td>73.6 -0.00272 0.00008</td>
<td>-0.1</td>
<td>-0.2%</td>
</tr>
<tr>
<td>-2100 131.250 0</td>
<td>9.7 -0.00350 0.00075</td>
<td>11.0 -0.00304 0.00071</td>
<td>1.3</td>
<td>13.9%</td>
</tr>
<tr>
<td>-2250 140.625 0</td>
<td>15.0 -0.00350 0.00063</td>
<td>16.4 -0.00307 0.00061</td>
<td>1.4</td>
<td>9.3%</td>
</tr>
<tr>
<td>-2500 156.250 0</td>
<td>24.3 -0.00350 0.00048</td>
<td>25.7 -0.00308 0.00048</td>
<td>1.4</td>
<td>5.9%</td>
</tr>
<tr>
<td>-2750 171.875 0</td>
<td>34.1 -0.00350 0.00038</td>
<td>35.5 -0.00308 0.00039</td>
<td>1.4</td>
<td>4.2%</td>
</tr>
<tr>
<td>-3000 187.500 0</td>
<td>44.0 -0.00350 0.00031</td>
<td>45.5 -0.00307 0.00032</td>
<td>1.4</td>
<td>3.3%</td>
</tr>
<tr>
<td>-3250 203.125 0</td>
<td>54.1 -0.00350 0.00025</td>
<td>55.5 -0.00307 0.00026</td>
<td>1.4</td>
<td>2.6%</td>
</tr>
<tr>
<td>-3500 218.750 0</td>
<td>64.3 -0.00350 0.00021</td>
<td>65.8 -0.00306 0.00022</td>
<td>1.4</td>
<td>2.2%</td>
</tr>
<tr>
<td>-3750 234.375 0</td>
<td>74.6 -0.00350 0.00017</td>
<td>76.0 -0.00306 0.00019</td>
<td>1.4</td>
<td>1.9%</td>
</tr>
</tbody>
</table>

**Reinforcement design of concrete sections based on the arc-length method**
Table 7
Doubly-reinforced cross-section subjected to compression and uniaxial bending (ε_y = b_y / 2)

| Nx  | M_y | M_z | As tot | ε_c min | ε_s max | As tot | ε_c min | ε_s max | ΔA (cm²) | ΔA/As tot |
|-----|-----|-----|--------|--------|---------|--------|--------|---------|----------|-----------|-----------|
| 93.75 | 12.1 | -0.00350 | 0.00214 | 12.6 | -0.00344 | 0.00248 | 0.4 | 3.4% |
| 125.00 | 24.9 | -0.00350 | 0.00148 | 24.7 | -0.00350 | 0.00161 | -0.2 | -0.9% |
| 156.25 | 39.1 | -0.00350 | 0.00120 | 38.9 | -0.00350 | 0.00129 | -0.2 | -0.6% |
| 187.50 | 53.8 | -0.00350 | 0.00105 | 53.6 | -0.00339 | 0.00111 | -0.2 | -0.5% |
| 218.75 | 68.7 | -0.00350 | 0.00095 | 68.5 | -0.00344 | 0.00101 | -0.2 | -0.3% |

Table 8
Doubly-reinforced cross-section subjected to pure bending

| Nx  | M_y | M_z | As tot | ε_c min | ε_s max | As tot | ε_c min | ε_s max | ΔA (cm²) | ΔA/As tot |
|-----|-----|-----|--------|--------|---------|--------|--------|---------|----------|-----------|-----------|
| 93.75 | 12.1 | -0.00350 | 0.00214 | 12.6 | -0.00344 | 0.00248 | 0.4 | 3.4% |
| 125.00 | 24.9 | -0.00350 | 0.00148 | 24.7 | -0.00350 | 0.00161 | -0.2 | -0.9% |
| 156.25 | 39.1 | -0.00350 | 0.00120 | 38.9 | -0.00350 | 0.00129 | -0.2 | -0.6% |
| 187.50 | 53.8 | -0.00350 | 0.00105 | 53.6 | -0.00339 | 0.00111 | -0.2 | -0.5% |
| 218.75 | 68.7 | -0.00350 | 0.00095 | 68.5 | -0.00344 | 0.00101 | -0.2 | -0.3% |
The examples in Tables 2, 4, and 6, which consider combined compression and bending with smaller eccentricity, yield significant relative differences. A relative difference of -9.0% is found for C15 concrete (Table 4). The negative sign means that the parabola-rectangle diagram is more conservative. C30 and C45 concretes yield relative differences of 13.9% and 28.7%, respectively (Table 6). The positive sign means that the Sargin curve requires more reinforcement. As the absolute differences for C15, C30 and C45 are limited to -1.4 cm², 1.4 cm², and 3.6 cm², respectively, the relative differences are relevant for low reinforcement ratios.

Table 9 investigates reinforcement design in pure bending without compression reinforcement. The concrete class is C30. This analysis demonstrates the good convergence of the proposed method even without any contribution from steel to the stiffness of the compressive block. The relative differences are less than 1% in the first examples, when the tension reinforcement reaches the yield point ($\varepsilon_{s_{\text{max}}} \geq 0.00207$). In the last example, the relative difference is 5.7% and the reinforcement strain is below the yield point. ABNT NBR 6118:2014 [10] recommends compression reinforcement in beams to avoid a neutral axis in domain 4. The comparison between the same examples with and without compression reinforcement (Tables 8 and 9) shows that this recommendation also improves the correspondence between the parabola-rectangle and Sargin results in pure bending.

Figure 9 examines an example for the Sargin curve in Table 2 ($e_z = b_z /4, f_{ck} = 30 \text{ MPa}$ and $A_{s_{\text{tot}}} = 67.0 \text{ cm}^2$). The modulus of the stress resultant vector $|S|$ is plotted as a function of the modulus of the generalized strain vector $|k|$. The maximum strength value is obtained for $|k|=0.00455$, which corresponds to $\varepsilon_{c_{\text{min}}} = -0.00308$, 5.7%.
The required reinforcements are $A_{\text{PAR-RECT}} = 70.54 \, \text{cm}^2$ and $A_{\text{SARGIN}} = 74.56 \, \text{cm}^2$, respectively. The corresponding level arms are $z_s \text{PAR-RECT} = 0.524 \, \text{m}$ and $z_s \text{SARGIN} = 0.510 \, \text{m}$, respectively. Reinforcing bars do not reach the yield point in either case. The parabola-rectangle diagram and the Sargin curve yield $\sigma_{\text{c top}} = \sigma_{\text{c min}}$ and $|\sigma_{\text{c top}}| < |\sigma_{\text{c min}}|$, respectively, where $\sigma_{\text{c top}}$ is the stress at the top of the section and $\sigma_{\text{c min}}$ is the minimum compressive stress in the concrete. The concrete and steel force resultants are $R_x = R_y = 2003.73 \, \text{kN}$ and $R_z = 2057.53 \, \text{kN}$ for the parabola-rectangle and Sargin curves, respectively.

### 7. Conclusion

The reinforcement design of concrete sections based on the parabola-rectangle diagram is a practical and well-established model. However, the initial modulus of elasticity and plastic range of the parabola-rectangle diagram do not represent the actual behavior of concrete. Stress-strain relationships that better characterize concrete properties are needed for global limit analyses of concrete structures that consider their physical and geometric non-linear behavior. The Sargin curve is selected because it is a function of the peak point and initial modulus of elasticity and represents the descending branch of the stress-strain relationship.

This research proposes a numerical procedure for the reinforcement design of concrete sections that uses an arc-length method and yields good convergence in the descending branch of the Sargin curve, without having to consider the distributions of strain limits around pivot C in domain 5. Strain limits for concrete and steel are not required, but they are included in order to avoid excessive deformation. The parabola-rectangle and Sargin curves are considered by using the code provisions for cross-sections and global limit analyses, respectively. The reinforcement design using the parabola-rectangle diagram is based on the section model in ABNT NBR 6118: 2014 [10]. The Sargin curve is implemented according to the global nonlinear model in CEN Eurocode 2: 2004 [8]. The examples consider characteristic concrete strength values $\bar{c}$ of 15, 30, and 45 MPa. The typical $0.25 \, \text{m} \times 0.85 \, \text{m}$ rectangular section and global analysis parameters for the parabola-rectangle and Sargin curves are not necessarily conservative when compared to the parabola-rectangle diagram.

Despite the good correspondence observed in most examples, the investigation shows that the results of the Sargin curve are not necessarily conservative when compared to the parabola-rectangle diagram. For this reason, a global limit analysis using the Sargin curve still requires the analysis of all cross-sections with the parabola-rectangle diagram.

The proposed reinforcement design method is efficient, numerically robust, and capable of considering other stress-strain relationships with or without descending branches. The examples use local and global analysis parameters for the parabola-rectangle and Sargin curves, respectively. The validation of a single calculation model for section and global limit analyses could motivate future investigations.

### 8. Acknowledgements

The first author thanks the Coordination for the Improvement of Higher Education Personnel (CAPES) for financial support. The authors thank the valuable suggestions of Prof. Benjamin Ernani Diaz.

### 9. References


Reinforcement design of concrete sections based on the arc-length method


